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TECHNICAL MEMORANDUM 1238

TWO-DIMENSIONAL WING THEORY IN THE SUPERSONIC RANGE

By H. Hönl

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Abstract: The plane problem of the vibrating airfoil in supersonic flow is dealt with and solved within the scope of a linearized theory by the method of the acceleration potential.

Outline:

The present investigation was carried out by Professor Hönl of the Kaiser-Wilhelm-Institute for Flow Research by order of the Aerodynamic Experimental Station Göttingen, Institute for Unsteady

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I. INTRODUCTION

The present treatise deals with the plane problem of the vibrating airfoil in supersonic flow within the scope of a linearized theory. The method is based on the application of results pertaining to the field of moving sound sources[1] and, in addition, on the introduction of the acceleration potential for the movement of the air particles. In view of the latter the method can be regarded as a direct extension of Prandtl's "General Considerations on the Flow of Compressible Fluids"[2] to the unsteady problem of the vibrating wing; the following makes use of all fundamental ideas of this theory. This extension leads to an integro-differential equation for the strength of the doublet distribution which represents the vibrating airfoil; this equation which is characteristic of the mathematical development of the theory, is solved by means of a Laplace transformation. The use of improper functions (first and second derivatives of discontinuous functions) which has become familiar in the modern quantum theory, may appear to be an innovation, but the fundamental idea which is expressed in this formal approach is nothing but the interpretation known from the theory of characteristics of hyperbolic differential equations, namely that disturbances and therefore also discontinuities in the hyperbolic case are propagated along real characteristics, in the present case notably on the boundary planes of the Mach wedge. Altogether the author believes to have found here a procedure which leads very simply and directly to a solution in the case of the vibrating airfoil moved in translation at supersonic speed; it can probably be profitably applied to similar problems as well.

Dr. L. Schwarz has solved the present problem in another way; I am particularly indebted to him for several critical observations which benefited my work. Apart from differences in the notation there is complete agreement between Schwarz' and my results so that their correctness ought to be fairly certain. I should like to mention that the theory has been carried through only to where it merges into the general final formulas of Schwarz. All further statements would be identical with Schwarz' statements and could not offer anything new.

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[1] H. Hönl, Ann.d.Phys. (now being printed) [1].
[2] L. Prandtl, Lufo 13 (1936) p. 313 [2] TM 805.

II. STATEMENT OF THE PROBLEM

As first noted by Lanchester and carried further by L. Prandtl,[3] the introduction of an acceleration potential ψ from which the acceleration vector $\vec{b}$

$$\vec{b} = \frac{Dw}{dt} = \frac{\partial w}{\partial t} + (w, \nabla w) \tag{1}$$

[w is used for German script w]

of the air particles (w velocity vector) is derived according to $\vec{b}$ = grad ψ, proves to be especially useful for the treatment of the wing in a compressible medium. ψ then results according to Euler's hydrodynamic equations for frictionless liquids, apart from an arbitrary constant, as identical with the negative pressure function, thus

$$\psi + \int \frac{dp}{\rho} = \text{Constant} \tag{2}$$

The fundamental idea in applying the acceleration potential to the wing theory in the shape given to it by Prandtl is the following: If the perturbation velocities are linearized, ψ, on the one hand, satisfies the well known equation of wave propagation

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{3}$$

(c sonic velocity); on the other hand, however, ψ experiences according to (1) a discontinuous change in penetrating the lifting surface on which there appears a pressure jump between the lower and upper surfaces. These conditions suggest the use, in determining ψ, of analogies from electrostatics (for the steady-flow problem) or electrodynamics (for the unsteady-flow problem).

We will limit ourselves to the <u>two-dimensional</u> problem of the infinitely long wing and suppose that the lifting surface is a strip of width l, located in the x,z-plane, parallel to the z-axis.

Furthermore, it will be useful to employ a system of coordinates which is attached to the airfoil so that the wing in reference to

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[3] L. Prandtl, elsewhere.

this coordinate system executes small vibrations but has no translation; if one assumes that the wing in the original coordinate system (medium at rest) moves along the negative x-axis with the velocity U, the wing in the new coordinate system (without translation) is in a flow of the same velocity in the positive x-axis, and equation (3) must be replaced by

$$\Delta\psi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\psi = 0 \qquad (3')$$

In this equation, $\partial/\partial t$ is to be left out in the case of steady flow. For the problems of unsteady flow we will limit ourselves to the case of the harmonically vibrating (infinitely thin) wing and accordingly make the substitution

$$\psi = \varphi(x,y)\, e^{-i\omega t} \qquad (4)$$

The wave equation (3') changes into

$$(M^2 - 1)\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \left(k^2 + 2ikM\frac{\partial}{\partial x}\right)\varphi = 0 \qquad (5)$$

with

$$k = \frac{\omega}{c}, \qquad M = \frac{U}{c} \qquad (5a)$$

(k quantity propagated, M Mach number)

We now assume that $\Phi(x,y)$ is a solution of the wave equation (5) - we call it an elementary solution - which corresponds to a sonic point source located at $x = y = 0$.[4]

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[4] H. Hönl, elsewhere.

Then obviously the "source potential"

$$\Phi_Q = U \int_0^l \Phi(x - \xi, y)\, f(\xi)\, d\xi$$

with an arbitrary source distribution $U\ f(x)$ along the wing chord $0 \leqq x \leqq l$ anywhere outside of the lifting surface will satisfy the equation (5) and the same is valid for

$$\varphi = \frac{\partial \Phi_Q}{\partial y} = U \int_0^l \Phi_y(x - \xi, y)\, f(\xi)\, d\xi \tag{6}$$

(The factor U is added to avoid unnecessary factors in the following formulas). The latter expression obviously represents apart from the time factor $e^{-i\omega t}$ the retarded potential of a doublet distribution; $f(\xi)$ is now a measure for the strength of the doublet distribution and is, therefore, proportional to the jump in φ while penetrating the surface. If we substitute (6) into (4), ψ will meet all requirements which are to be demanded of the acceleration potential and can therefore be identified with it. However, it must be stressed that ψ and φ will be uniquely determined for the whole space only when the source and vortex distribution is given everywhere and in addition a boundary condition or ray condition[5] at infinity is prescribed for φ. In fact, a vortex wake in general adjoins the trailing edge of the wing which also will contribute to the acceleration potential. The conditions for the case of supersonic flow are rendered especially simple by the fact that each source or vortex element takes effect downstream only and that therefore the air forces on the wing are solely determined by the doublet distribution replacing the pressure jump. Concerning the boundary condition at infinity we shall have to require that φ and therefore the perturbation velocity components u and v disappear outside of the Mach wedge which starts at the leading edge of the wing (therefore, especially for $x < 0$) but in the infinite inner region of the Mach wedge satisfy a ray condition; these requirements are automatically met by our equation (6) according to the definition of the elementary solution.

Therefore, our task will be to find conditions which uniquely determine the source distribution $f(\xi)$ in (6). One such condition

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[5] Compare A. Sommerfeld, Jahresber. d. DMV 21, pp 309–353 (1913) [4].

results from the fact that the downwash distribution $w(x, t) = v_0(x)\, e^{-i\omega t}$ on the wing is given. Therefore, we further need at first relations between the acceleration potential Ψ and the perturbation velocities u, v (u horizontal, v perpendicular) which permit the expression of the latter by Ψ within the scope of the linearized theory[6]. Thus we obtain corresponding to our assumption about the establishment of the coordinate system according to (1) apart from terms of the second order in the perturbation velocities

$$\vec{b} = \frac{\partial \mathfrak{w}}{\partial t} + U\frac{\partial \mathfrak{w}}{\partial x} \tag{7}$$

and, therefore, by substituting $\vec{b} = \operatorname{grad} \Psi$, $\mathfrak{w} = \operatorname{grad} \Phi$

$$\Psi = \frac{\partial \Phi}{\partial t} + U\frac{\partial \Phi}{\partial x} + \chi(t) \tag{7'}$$

where $\chi(t)$ stands for an arbitrary function of time. Since we will assume, however, that no change shall take place at infinity, we can immediately specialize $\chi(t) = C$. If we assume $\Psi(x,y;\ t)$ as given there follows now for the velocity potential[7]

$$\Phi = \frac{1}{U}\int_{-\infty}^{x} \Psi\left(x',y;t - \frac{x - x'}{U}\right) dx' + \frac{C}{U}x + D \tag{8}$$

with D as a further (insignificant) integration constant. In this equation the integral appearing in the first term

$$\bar{\Phi} = \frac{1}{U}\int_{-\infty}^{x} \Psi\left(x',y;t - \frac{x - x'}{U}\right) dx' \tag{8'}$$

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[6] Compare H.G. Küssner, Lufo 17, p. 370, Abschn.4 [5].

[7] An arbitrary function $f(t - x/U)$ to be added in (8) according to (7') is reduced to a constant since otherwise the wave equation for Φ (analogous to (3) and (5), respectively) would not be satisfied.

corresponds to the potential of the perturbation velocities, the part containing x linearily corresponds to a superposed undisturbed flow in the x-direction; therefore one has to equate $C = U^2$. From (8') follows

$$v = \frac{\partial \tilde{\psi}}{\partial y} = \frac{1}{U} \int_{-\infty}^{x} \frac{\partial \psi}{\partial y}\left(x', y;\ t - \frac{x - x'}{U}\right) dx' \tag{9}$$

as well as a further integral for u not needed at present.

If we make use of (4) for harmonic vibrations we obtain for the x-axis:

$$v = v_o(x)\ e^{-i\omega t} = \underset{y \to o}{\text{Limes}}\ \frac{e^{-i\omega t}}{U} \int_{-\infty}^{x} \varphi_y(x', y)\ e^{i\frac{\omega}{U}(x-x')}\, dx' \tag{10}$$

or

$$e^{-i\frac{\omega}{U}x}\ v_o(x) = \underset{y \to o}{\text{Limes}}\ \frac{1}{U} \int_{-\infty}^{x} \varphi_y(x', y)\ e^{-i\frac{\omega}{U}x'}\, dx'$$

and finally by differentiation with respect to x:

$$e^{-i\frac{\omega}{U}x}\ \frac{d}{dx}\left[e^{-i\frac{\omega}{U}x}\ v_o(x)\right] = \underset{y \to o}{\text{Limes}}\ \frac{1}{U}\ \varphi_y(x,y) \tag{11}$$

We now have to substitute (6) on the right side of (11). Taking the fact into consideration that only those sources contribute to the downwash at the point x of the wing in supersonic flow which lie upstream with respect to x, for which therefore $\xi \leq x$, we can replace the fixed upper limit of the integral (6) by the variable x; as the lower limit we will select $-\infty$, establishing $f(\xi) = 0$ for $\xi < 0$, so that $f(\xi)$ for $\xi = 0$ will, in general, be discontinuous. In the limit $y \to 0$ the right side of (11) must approach the same limit value for both signs of $\eta = |y|$. Thus we obtain from (6) and (11), taking (5a) also into consideration,

$$\underset{\eta \to o}{\text{Limes}} \int_{-\infty}^{x} \Phi_{yy}(x - \xi,\ \pm\eta)\ f(\xi)\ d\xi = \frac{dv_o}{dx} - i\,\frac{k}{M}\,v_o \tag{12}$$

The right side of (12) is a given function of x; however, it must be taken into consideration that now $v_o(x)$ also at the point $x = 0$ is to be regarded as discontinuous $(v_o(x) = 0$ for $x < 0)$, so that $\frac{dv_o(x)}{dx}$ at $x = 0$ has to be represented by an improper function (δ-function). Correspondingly, in the derivative Φ_{yy} also singularities of higher order appear in the integrand. By the integral equation (12) with a given kernel Φ_{yy} and given right side the distribution function $f(x)$ is uniquely determined.

The calculation of the pressure jump on the wing which is the real goal of this paper, results after solution of the equation (12) immediately from (2). First, the change of the pressure function corresponding to the infinitesimal pressure change Δp will be

$$\Delta \int \frac{dp}{\rho} = \left(\frac{1}{\rho}\right)_o \Delta p = \frac{\Delta p}{\rho_o}$$

with ρ_o standing for the average density of the medium, and therefore, according to (2) the pressure jump π between lower and upper side of the wing (the indices + and - refer to upper and lower side):

$$\pi = \Delta p_- - \Delta p_+ = \rho_0(\psi_+ - \psi_-) \tag{13}$$ [a]

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[a] Equation (13) for the pressure jump π is essentially identical with Bernoulli's relation between pressure and velocity; this relation is represented within the scope of the linearized theory for the case of unsteady flow by the equation

$$\int \frac{dp}{\rho} + U\frac{\partial\psi}{\partial x} + \frac{\partial\tilde{\psi}}{\partial t} = \text{Constant} \tag{α}$$

or, using the notation of $u = \frac{\partial\tilde{\psi}}{\partial x}$, by

$$\frac{\pi}{\rho_0} = U(u_+ - u_-) + \left(\frac{\partial\tilde{\psi}}{\partial t}\right)_+ - \left(\frac{\partial\tilde{\psi}}{\partial t}\right)_- \tag{β}$$

The calculation of the horizontal component u yields according to (8) at first

$$u = \frac{\partial\tilde{\psi}}{\partial x} = \frac{1}{U}\psi(x,y;t) - \frac{1}{U^2}\int_{-\infty}^{x}\psi_t\left(x,y;t - \frac{x - x'}{U}\right)dx'$$

$$= \frac{1}{U}\left[\psi(x,y;t) - \frac{\partial\tilde{\psi}(x,y;t)}{\partial t}\right] \tag{γ}$$

therefore, according to (β) in agreement with (13)

$$\frac{\pi}{\rho_0} = \psi_+ - \psi_-$$

The explicit calculation of u and $\frac{\partial\tilde{\psi}}{\partial t}$ from $\tilde{\psi}$ according to (β) proves therefore to be a detour here.

Thus the solution of our problem is traced back to the solution of the integral equation (12) for the distribution function $f(x)$ from which π can be calculated according to (4), (6), and (13). Since singularities which originate in discontinuities of $v_0(x)$ and $\Phi(x, y)$ appear on the right as well as on the left side of (12), the solution of (12) shall be preceded in the following section by a discussion of the properties of the δ-function.

III. THE METHOD OF THE δ-FUNCTION

By the function $\delta(x)$ - also written $\delta(x-0)$ – we understand an improper function, which, with the exception of the points $x = 0$ becomes infinite in such a way that

$$\int_{x_1}^{x_2} \delta(x)\, dx = 1 \tag{14}$$

becomes valid if the interval $x_1 \ldots x_2$ contains the point $x = 0$ as an inner point. In order to attach a stricter mathematical meaning to the arithmetical operations making use of δ, it is necessary to consider $\delta(x)$ as limit of a sequence of continuous functions. We assume therefore at first continuous positive functions $\varphi_p(x)$ which have a steep maximum at the point $x = 0$ and are also subjected to the condition

$$\int_{x_1}^{x_2} \varphi_p(x)\, dx = 1 \tag{14'}$$

Let $\varphi_p(x)$ be a function of a parameter p in such a way that $\varphi_p(x)$ everywhere, with the exception of the point $x = 0$, approaches zero as p approaches a fixed limiting value, for instance $p = 0$. The "limit function" $\delta(x)$ thus characterized may be denoted as "prong function." All mathematical expressions containing the δ-functions in a symbolic way must, therefore, be considered as limiting processes in which at first $\delta(x)$ is replaced by the sequence of functions $\varphi_p(x)$ and finally the limit $p \rightarrow 0$ is taken.

After this comment there results immediately the property of the δ-function which is most important for its application, the integral relation

$$\int_{x_1}^{x_2} \delta(x - a)\, F(x)\, dx = F(a) \tag{15}$$

for $x_1 < a < x_2$, in case $F(x)$ at the point $x = a$ is continuous.

In addition, we will consider the sequence of the derivatives $\varphi_p'(x)$ of the function $\varphi_p(x)$ defined above. We call the limit function characterized by this sequence $\delta^{(1)}(x)$; therefore, we may equate symbolically:

$$\delta^{(1)}(x) = \frac{d\delta(x)}{dx} \tag{16}$$

Since we may transform integrals, which contain in the integrand the continuous functions $\varphi_p'(x)$, by integration by parts into integrals which contain $\varphi_p(x)$ instead of $\varphi_p'(x)$, there follows from (15) also

$$\int_{x_1}^{x_2} \delta^{(1)}(x - a)\, F(x)\, dx = F'(a) \tag{17}$$

where $x_1 < a < x_2$, for all functions $F(x)$ which have a continuous derivative at the point $x = a$; use is made of the fact that the functions $\varphi_p(x)$ disappear in the limit $p \rightarrow 0$ at the limits of the interval.

Finally we mention that the sequence of functions $\varphi_p(x)$ may also be regarded as a sequence of the derivatives of function $g_p(x)$ which approximates the "step function"

$$s(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqq 0 \end{cases} \tag{18}$$

in a continuous way. In this sense we may then substitute symbolically, in analogy to (11),

$$\delta(x) = \frac{ds(x)}{dx} \tag{19}$$

The application of the method of the δ-function to the present problem is based on the fact that the elementary solution $\Phi(x, y)$ of the wave equation (5) is, for supersonic conditions, quite generally a <u>discontinuous</u> function - namely zero outside of the Mach wedge, but $\neq 0$ inside of it and also on the boundary when approached from the inside so that the first and second derivatives $\Phi_y(x, y)$ and $\Phi_{yy}(x, y)$ which appear under the integral according to (6) and (12) lead to the separation of improper function parts on the boundaries of the Mach wedge. On the other hand, the performance of the prescribed integrations according to (15) and (17) does not present any difficulties as long as we may suppose the distribution function $f(x)$ and its derivative $f'(x)$ everywhere as <u>continuous</u>. This assumption is fulfilled for the extent of the wing $0 < x < l$ with exception of its leading and trailing edge as well as in front of and behind the wing, that is, for $x < 0$ and $x > l$, where $f(x) = 0$. The points $x = 0$ and $x = l$ are exceptions. We will try to evade this difficulty by assuming at first that $f(x)$ is continuous also near the points $x = 0$ and $x = l$ and will proceed only <u>subsequently</u> to the limiting case of a discontinuous distribution function. This procedure always leads to a uniquely determined result. The mathematical meaning is that one has to proceed <u>first</u> to the limit $p \rightarrow 0$ and only subsequently to the limiting case of a discontinuous distribution function or downwash distribution on the wing (right side of (12)) since the performance of the integrations (6) and (12) already involves the application of the δ-symbol. Such a procedure that regards the compression shock along the boundary of the Mach wedge which corresponds to the elementary solution $\Phi(x, y)$ as infinitely sharp (linearized theory!) in contrast with the always incompletely realized sharp limitation of the wing, is probably fully justified from the physical point of view.

IV. THE PLATE AT A GIVEN ANGLE OF ATTACK IN SUPERSONIC FLOW (STEADY-FLOW PROBLEM)

We illustrate the method for calculation of the air forces at first by the steady flow - example of the plate at rest at an

infinitesimal angle β. We shall obtain again results already known.[9]

In this example the downwash is constant at both sides of the plate, (we equate $v_o = -b$), but zero in front of and behind the plate. Obviously $\beta = \frac{b}{U}$ (apart from quantities of a higher order) (See fig. 1.) If we now form the derivative $\frac{dv_o}{dx}$, as provided for in (12), this quantity will become at $x = 0$ and $x = l$, respectively, $-\infty$ and $+\infty$, but in such a way that the function

$$v_o(x) = \int_{-\infty}^{x} \frac{dv_o(\xi)}{d\xi} d\xi = \begin{cases} 0 & \text{for } x < 0 \\ -b & \text{for } 0 < x < l \\ 0 & \text{for } x > l \end{cases} \tag{20}$$

at $x = 0$ and $x = l$ experiences a discontinuity by $\mp b$. According to (19) we, therefore, may substitute for the right side of (12) - taking into consideration that the second term of the right side disappears in the steady-flow case ($\omega = 0$ or $k = 0$, respectively) -

$$\frac{dv_o}{dx} = -b\delta(x - 0) + b\delta(x - l) \tag{20'}$$

On the other hand, let us consider the elementary solution $\Phi(x, y)$ appearing in (12) and its derivatives. By $\Phi(x, y)$ we understand in the steady-flow case a discontinuous function which assumes inside of the Mach wedge

$$\left|\frac{y}{x}\right| = \tan \alpha, \qquad x > 0; \qquad \tan \alpha = \frac{1}{\sqrt{M^2 - 1}} \tag{21}$$

with the apex $x = y = 0$ and at half the opening angle α a constant value (that is, a value independent of x, y and with a certain normalization dependent only on M), but disappears outside the Mach wedge.[10] It does not mean a restriction of universal validity

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[9] Compare Ackeret, Zs.f. Flugtechn. Motorluftsch. Bd. 16 (1925), p.72 [6].

[10] Compare L. Prandtl, elsewhere. Equation (30).

for our problem if we equate the value of the function inside the Mach wedge = 1. Introducing the step function (18) obviously makes

$$\Phi(x,y) = \begin{cases} s(x \mp \cot\alpha\, y) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{22}$$

with the upper sign valid for $y > 0$, the lower sign valid for $y < 0$. If we further take (19) and (16) into consideration, we obtain for $x > 0$ with the abbreviation $q = x \mp \cot\alpha\, y$:

$$\Phi_y(x,y) = \mp \cot\alpha\, \Phi'(q) = \mp \cot\alpha\, \delta(q) \tag{23a}$$

$$\Phi_{yy}(x,y) = \cot^2\alpha\, \Phi''(q) = \cot^2\alpha\, \delta^{(1)}(q) \tag{23b}$$

If we now apply the integral relation (17) the integral on the left side of (12) becomes, if we use again $\eta = |y|$ instead of y,

$$\int_{-\infty}^{x} \Phi_{yy}(x - \xi, \pm\eta)\, f(\xi)\, d\xi = \cot^2\alpha \int_{-\infty}^{x} \delta^{(1)}(x - \xi - \cot\alpha\, \eta)\, f(\xi)\, d\xi$$

$$= \cot^2\alpha\, f'(x - \cot\alpha\, \eta) \tag{24}$$

Therewith we obtain from (12) with the help of (20') in the limit $\eta \longrightarrow 0$ the differential equation

$$\cot^2\alpha\, f'(x) = -b\delta(x - 0) + b\delta(x - l) \tag{25}$$

from which we conclude that $f(x)$ must be discontinuous at the points $x = 0$ and $x = l$; and so, apart from a constant, the function

$$f(x) = as(x - 0) - as(x - l) \tag{26}$$

is, according to (19), the solution of (25), if a is selected:

$$a = \frac{-b}{\cot^2\alpha} \tag{26a}$$

The constant, which is as yet undetermined, is obtained from the fact that $f(x)$ must always vanish <u>ahead of</u> the leading edge $(x \leq 0)$ and is, therefore, left out in (26). Of course, we likewise put $f(x) = 0$ <u>behind</u> the trailing edge $(x > l)$, a condition which is then automatically fulfilled according to (27). However, there is obviously only <u>one</u> integration constant at our disposition for the transition from (25) to (26). The reason for the possibility of giving the solution of (25) in the form (26) is, according to (20), that the downwash v_o performs an equal and opposite jump at $x = 0$ and $x = l$ so that, behind the trailing edge $v_o = 0$. Therewith, we only confirm the correctness of our original assumption. For non-plane lifting surfaces or for the problem of unsteady flow the above statements will, in general, be no longer valid; a vortex-wake will then adjoin the trailing edge

We calculate the horizontal component u of the perturbation velocity. From (8') and (4) one obtains for the steady-flow case

$$u = \frac{\partial\bar{\Psi}}{\partial x} = \frac{\psi}{U} = \frac{1}{U}\,\varphi(x,y) \tag{27}$$

and further, according to (6),

$$u = \int_{-\infty}^{x} \Phi_y(x - \xi, \pm\eta)\, f(\xi)\, d\xi \tag{27a}$$

if we fix the integration limits as in (12). If we go back to (22) and (23a) and use equation (15) for the transformation of the integral, we obtain, because of (26) and (26a),

$$u = \mp\cot \alpha \int_{-\infty}^{10} \delta(x - \xi - \cot \alpha\eta)\, f(\xi)\, d\xi$$

$$= \mp\cot \alpha f(x - \cot \alpha\eta)$$

$$= \pm \frac{b}{\cot \alpha} \left[s(x - \cot \alpha\eta) - s(x - l - \cot \alpha\eta) \right]$$

and in the limit $\eta \longrightarrow 0$:

$$u_o = \pm \frac{b}{\cot \alpha} \left[s(x - 0) - s(x - l) \right] \tag{28}$$

If we compare (28) with (20) for which we can also write

$$v_o = -b \left[s(x - 0) - s(x - l) \right] \tag{20a}$$

we find

$$u_o = \mp v_o \tan \alpha \tag{29}$$

(In all formulas the upper sign always refers to the upper side, the lower one to the lower side of the lifting surface.)

The pressure difference π on both sides of the lifting surface is now immediately obtained from (13), (27), and (29). There follows with $v_o = -b$:

$$\pi = 2\rho_o U b \tan \alpha \tag{30}$$

and from this equation the lift

$$A = \pi l = 2\rho_0 U b l \tan \alpha \tag{30a}$$

In order to obtain the lift coefficient c_a we have to divide by $\frac{\rho_0}{2}U^2 l$. If we replace b by the angle of attack $\beta = \frac{b}{U}$, there follows from (30a)

$$c_a = 4\beta \tan \alpha \tag{31a}$$

The result for the drag coefficient c_w will be, since the suction force at the leading edge is eliminated in supersonic flow,

$$c_w = \beta c_a = 4\beta^2 \tan \alpha \tag{31b}$$

The formulas (31a) and (31b) are in agreement with the results of Ackeret who for the first time calculated the air forces on the plate at a given angle of attack in supersonic flow in 1925.

As a conclusion to this treatment we want to stress once more that the velocity and pressure distribution of the plate at a given angle of attack were traced back essentially to the discontinuities of the distribution function $f(x)$. This fact is expressed clearly in the connection between the formulas (26)and (28). Thus a discontinuity surface of the u-distribution as well as the v-distribution spreads from the leading edge of the lifting surface which coincides with the boundary planes of the Mach wedge and causes an expansion wave in the upper region $y > 0$ and a compression wave in the lower region $y < 0$ (fig. 2). The effect of this discontinuous change in velocity is in our example exactly cancelled by the second discontinuity surface starting from the trailing edge $x = l$ (compression wave for $y > 0$, expansion wave for $y < 0$) so that the flow behind the second Mach wedge which starts at the trailing edge is again undisturbed. Since obviously the sphere of action of a discontinuity surface of $f(x)$ in supersonic flow extends downstream only, there follows that the problem of pressure distribution on the wing in general is already uniquely determined without any additional "flow-off condition" (requirements for finiteness for the v-distribution immediately behind the trailing edge). The discontinuity of $f(x)$ at $x = l$ involves in general the appearance of vortices at the

trailing edge which, however, need not be ascertained for the investigation of the air forces on the wing.

V. BENDING OSCILLATIONS OF A PLANE WING AT ZERO ANGLE OF ATTACK

SETTING UP OF AN INTEGRO-DIFFERENTIAL EQUATION

The bending oscillations of a plane wing at zero angle of attack represent the simplest problem of unsteady flow. Let y_o be the (infinitesimal) displacement of the mean camber line from its zero position (x-axis); we then substitute

$$y_o = Ae^{-i\omega t} \quad \text{for} \quad 0 \leqq x \leqq l \tag{32}$$

Then the downwash in the same interval becomes

$$w(x,t) = v_o(x)\ e^{-i\omega t} = \frac{\partial y_o}{\partial t} = -i\omega Ae^{-i\omega t}$$

and, therefore,

$$\frac{dv_o}{dx} = -i\omega A\delta(x - 0) + \ldots \tag{33}$$

By ... we refer to the contribution of the second discontinuity of v_o at $x = l$ which, however, does not explicitly enter our calculations since we can break off the wing at an arbitrary point $x = l$ without having to change the results for $x < l$.

Our most important problem now is the transformation of the integral on the left side of equation (12) and the transformation of this equation into an integro-differential equation for f of the ordinary type. The integral equation (12) has as in the problem of steady flow an essentially degenerate kernel since the elementary solution Φ on the limit of the Mach wedge becomes discontinuous by itself; its derivatives, therefore, are to be represented by the improper functions δ and $\delta^{(1)}$. Therefore we separate $\Phi_y(x, y)$ and $\Phi_{yy}(x, y)$ each into a degenerate part due to the discontinuity of the function Φ at the boundary of the Mach wedge and into a

regular part due to the analytic behavior of Φ in the inner region of the Mach wedge. The elementary solution $\Phi(x, y)$ of (5) for the time-periodic case of a sonic point source can be taken from a former investigation.[11] We normalize the function Φ in such a way that the jump of Φ in penetrating the Mach wedge has exactly the amount 1; the elementary solution then reads:

$$\Phi(x,y) = \begin{cases} e^{i\frac{M}{\epsilon}kx} J_0\left(\frac{k}{\sqrt{\epsilon}}\sqrt{\frac{x^2}{\epsilon} - y^2}\right) & \text{inside the Mach wedge} \\ 0 & \text{outside the Mach wedge} \end{cases} \tag{34}$$

where we further introduce for abbreviation

$$\epsilon = \cot^2 \alpha = M^2 - 1 \tag{35}$$

Taking the discontinuity of Φ on the boundary planes $x - \cot \alpha |y| = 0$ of the Mach wedge (for $x > 0$) into consideration, we obtain

$$\Phi_y(x,y) = \mp\sqrt{\epsilon}\ \delta\left(x - \sqrt{\epsilon}\,\eta\right) e^{i\frac{M}{\epsilon}kx} + \Phi_y^*(x,y) \tag{36a}$$

$$\Phi_{yy}(x,y) = \epsilon\delta^{(1)}\left(x - \sqrt{\epsilon}\,\eta\right) e^{i\frac{M}{\epsilon}kx} + \Phi_{yy}^*(x,y) \tag{36b}$$

where $\Phi^*(x, y)$ characterizes the part of $\Phi(x,y)$ which is, within the Mach wedge, continuous (and analytic); outside of the Mach wedge the derivatives disappear. With the insertion of (36b) into the integral on the left side of (12), the first part of (36b) yields by integration by parts

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[11] H. Hönl, elsewhere. Equation (40)

$$\epsilon\int_{-\infty}^{x}\delta^{(1)}(x-\xi-\sqrt{\epsilon}\,\eta)\,e^{i\frac{M}{\epsilon}k(x-\xi)}\,f(\xi)\,d\xi$$

$$=+\epsilon\int_{-\infty}^{x}\delta(x-\xi-\sqrt{\epsilon}\,\eta)\,\frac{d}{d\xi}\left[e^{i\frac{M}{\epsilon}k(x-\xi)}\,f(\xi)\,d\xi\right]$$

$$=\epsilon\left\{\left[-i\frac{M}{\epsilon}k\,f(\xi)+f'(\xi)\right]e^{i\frac{M}{\epsilon}k(x-\xi)}\right\}_{\xi=x-\sqrt{\epsilon}\,\eta}$$

and therefore in the limit $\eta \longrightarrow 0$:

$$-iMkf(x)+\epsilon f'(x)$$

The integral equation (12) therefore assumes, on account of (33), the form

$$-iMkf(x)+\epsilon f'(x)+\int_{-\infty}^{x}\Phi^{*}_{yy}(x-\xi,0)\,f(\xi)\,d\xi$$

$$=\begin{cases}-i\omega A\left[\delta(x-0)-i\frac{k}{M}\right] & \text{for } x \gtreqless 0\\ 0 & \text{for } x<0\end{cases} \qquad (37)$$

The right side of (37) shows that $f(x)$ in any case must be discontinuous at $x = 0$, so that $f'(x)$ contains an improper function $\delta(x - 0)$ as a constituent part that with proper choice of a multiplying constant will cancel out against the right side. This is natural since the function $f(x)$ for $x < 0$, that is, ahead of the wing, must vanish together with its derivative $f'(x)$ (so that (37) for $x < 0$ is solved by $f(x) = 0$), and therefore will, in general, show unsteady behavior at the front boundary of the wing, as already shown in the example of steady flow (IV). If we, therefore, set up the equation

$$f(x)=as(x-0)+f_1(x) \qquad (38)$$

where a represents a constant yet to be determined and the

function $f_1(x)$ defined for $x \geqq 0$ has to satisfy the condition that it changes continuously into

$$f_1(0) = 0 \tag{38a}$$

(37) becomes:

$$\begin{aligned} &- iMka - iMkf_1(x) + \epsilon\left[a\delta(x - 0) + f_1'(x)\right] \\ &+ a\int_0^x \Phi_{yy}(x - \xi, 0)\, d\xi + \int_0^x \Phi^*_{yy}(x - \xi, 0)\, f_1(\xi)\, d\xi \\ &\qquad = -i\omega A\left[\delta(x - 0) - i\frac{k}{M}\right] \end{aligned} \tag{39}$$

If we now put

$$a = \frac{-i\omega A}{\epsilon} \tag{40}$$

the δ-function will be completely eliminated from (39) and we shall obtain for $f_1(x)$ a <u>linear inhomogeneous</u> (Volterra) <u>integro-differential equation of the convolution type, of an</u> otherwise, regular character

$$\begin{aligned} &iMkf_1(x) + \epsilon f_1'(x) + \int_0^x \Phi^*_{yy}(x - \xi, 0)\, f_1(\xi)\, d\xi \\ &= a\left(iMk - \int_0^x \Phi^*_{yy}(\xi, 0)\, d\xi\right) - i\frac{k}{M}\epsilon a \end{aligned} \tag{41}$$

its solution is uniquely determined by the boundary condition (38a).

The solution of the integro-differential equation (41) required a further determination of its kernel. If we go back to (34) we obtain by differentiation for the inner region of the

Mach wedge:

$$\Phi_y^*(x,0) = 0 \qquad \Phi_{yy}^*(x,0) = \frac{k^2}{\epsilon}\, e^{i\frac{M}{\epsilon}kx}\, \frac{J_1\left(\frac{kx}{\epsilon}\right)}{\frac{kx}{\epsilon}} \tag{42}$$

where the equation

$$J_o'(z) = -J_1(z)$$

was made use of. If we introduce further the dimensionless variable (reduced length)

$$z = \frac{kx}{\epsilon} \tag{43}$$

instead of x, we can also write:

$$\Phi_{yy}^*(x,0) = \frac{k^2}{\epsilon}\, G(z) \ ; \qquad G(z) = e^{iMz}\, \frac{J_1(z)}{z} \tag{42a}$$

Equation (41) then assumes the form

$$-iMF_1(z) + F_1'(z) + \int_0^z G(z - \zeta)\, F_1(\zeta)\, d\zeta$$
$$= a\left[iM - i\frac{\epsilon}{M} - \int_0^z G(\zeta)\, d\zeta\right] \tag{44}$$

The symbol F_1, rather than f_1, was selected in this equation in view of the replacement by z of x. In the following section VI we are going to solve the integro-differential equation (44) for the boundary condition (38a) by means of a Laplace transformation; the result reads (for $z = 0$):

$$F_1(z) = a\left[e^{iMz}J_o(z) - i\frac{\epsilon}{M}\int_0^z e^{iMu}J_o(u)\, du - 1\right] \tag{45}$$

With this the distribution function $f(x)$ also is determined. By writing $F(z) \equiv aH(z)$ instead of $f(z)$, we obtain from (38)

$$F(z) \equiv aH(z) = \begin{cases} a\left[e^{iMz}J_o(z) - i\frac{\epsilon}{M}\int_0^z e^{iMu}J_o(u)\,du\right] & \text{for } z \geqq 0 \\ 0 & \text{for } z < 0 \end{cases} \tag{46}$$

where the value (40) has to be substituted for a.

In addition we shall prove that the pressure distribution on the wing for the general case of unsteady flow also is determined by the distribution function $f(x)$ or $F(z)$, respectively, alone. If, to this end, we go back to (4) and (6), we have at first

$$\psi(x,y;t) = e^{-i\omega t}\,U\int_{-\infty}^{x}\Phi_y(x-\xi,\ \pm\eta)\,f(\xi)\,d\xi$$

and from this equation according to (36a), because of the disappearance of $\Phi^*_y(x,y)$ in the limit $|y| \rightarrow 0$, (compare (42))

$$\begin{aligned}\psi(x,\pm 0;t) &= \mp\sqrt{\epsilon}\,e^{-i\omega t}\lim_{\eta \rightarrow 0}\int_{-\infty}^{x}\delta(x-\xi-\sqrt{\epsilon}\,\eta)\,e^{i\frac{M}{\epsilon}(x-\xi)}f(\xi)\,d\xi \\ &= \mp\sqrt{\epsilon}\,f(x)\,e^{-i\omega t}\end{aligned} \tag{47}$$

By substituting (47) into (13) we thus obtain, again replacing $f(x)$ by $F(z)$, for the pressure jump π on the wing

$$\pi = -2\rho_o\,U\sqrt{\epsilon}\,F(z)\,e^{-i\omega t} \tag{48}$$

This formula is valid in general. From it we obtain for the special case of bending oscillations, according to (46) and (40),

$$\pi = 2\rho_o\,U\,\frac{i\omega A}{\sqrt{\epsilon}}\,H(z)\,e^{-i\omega t} \tag{49}$$

by which the problem of the calculation of the pressure distribution on the wing is, for this special case, solved.

In discussing the equations (46), (47), and (49) one must not overlook the fact that the variable z contains the quantity ϵ, which decreases with $M \longrightarrow 1$ to zero, according to the equation $z = \frac{kx}{\epsilon}$. Therefore, the graphs for the F-, u-, and π-distribution as functions of the chord x in the limiting case $M \longrightarrow 1$ shrink more and more toward $x = 0$.

VI. SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION IN THE CASE OF AN ARBITRARY DOWNWASH DISTRIBUTION BY MEANS OF A LAPLACE TRANSFORMATION

Before we turn to the general problem named in the title we shall first derive the solution for the special case of the bending oscillations by means of a Laplace transformation. The formalism of the calculation can then be transferred without any difficulty of the general case of an arbitrary time-periodic downwash distribution.

(a) Bending Oscillations as a Special Case

Our problem is the solution of the integro-differential equation (44) for the boundary condition $F_1(0) = 0$ by means of a Laplace transformation.

If we denote the Laplace transformation $L_s\left[F_1(z)\right]$ by $\varphi(s)$, $L_s\left[G(z)\right]$ by $g(s)$, the equation which corresponds to (44) is linear in $\varphi(s)$ and becomes, taking into consideration the boundary condition $F_1(o) = 0$[12]

$$\left[s - iM + g(s)\right] \varphi(s) = a L_s \left[iM - i\frac{\epsilon}{M} - \int_0^z G(\zeta)\, d\zeta\right] = a\,\frac{iM - i\frac{\epsilon}{M} - g(s)}{s} \tag{50}$$

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[12] It is noteworthy that $L_s\left[F'(z)\right] = s\varphi(s) - F_1(0)$

compare G. Doetsch, Laplace transformation (J. Springer 1937), p. 153.

The Laplace transform of $J_1(z)/z$[13] is:

$$L_s\left[\frac{J_1(z)}{z}\right] = \int_0^\infty \frac{J_1(z)}{z} e^{-sz}\,dz = \sqrt{s^2 + 1} - s$$

and therefore, (compare (42a))

$$g(s) = \int_0^\infty \frac{J_1(z)}{z} e^{iMz} e^{-sz}\,dz = \sqrt{(s - iM)^2 + 1} - (s - iM) \quad (51)$$

The solution of (50) for $\varphi(s)$ is, therefore,

$$\varphi(s) = a\frac{iM - i\frac{\epsilon}{M} - g(s)}{s\left[s - iM + g(s)\right]} = a\frac{s - \sqrt{(s - iM)^2 + 1} - i\frac{\epsilon}{M}}{s\sqrt{(s - iM)^2 + 1}} \quad (52)$$

As is well known, the inverse transformation $\varphi(s) \longrightarrow F_1(z)$ is given by

$$F_1(z) = L_z^{-1}\left[\varphi(s)\right] = \frac{1}{2\pi i}\oint e^{sz}\varphi(s)\,ds \quad (53)$$

where the closed path of integration traversed in the positive sense encloses, in a unique sheet (schlichten Blatt) of the complex s-plane, the singularities of the integrand located in the finite part of the plane. If we finally introduce $t = s - iM$ as a new integration variable, $F_1(z)$ becomes, according to (52) and (53)

$$F_1(z) = a\frac{1}{2\pi i} e^{iMz}\oint e^{zt}\left[\frac{1}{\sqrt{t^2 + 1}} - \frac{1}{t + iM} - i\frac{\epsilon}{M}\frac{1}{(t + iM)\sqrt{t^2 + 1}}\right]dt$$

(54)

[13]G. Doetsch, p. 313 and 403.

One obtains the integral of the first term of the sum immediately from the integral representation for the Bessel function J_o :[14]

$$J_o(z) = \frac{1}{2\pi i} \oint \frac{e^{tz}}{\sqrt{t^2 + 1}} dt \tag{55}$$

the integral of the second term of the sum can be evaluated immediately, according to the residue theorem; for the integral of the third term of the sum one obtains immediately because of

$$\frac{\partial}{\partial z}\left[\frac{e^{iMz}}{2\pi i} \oint \frac{e^{tz}}{(t + iM)\sqrt{t^2 + 1}} dt\right] = \frac{1}{2\pi i} \oint \frac{e^{(iM+t)z}}{\sqrt{t^2 + 1}} dt = e^{iMz} J_o(z) \tag{55a}$$

the equation

$$\frac{e^{iMz}}{2\pi i} \oint \frac{e^{tz}}{(t + iM)\sqrt{t^2 + 1}} dt = \int_0^z e^{iMu} J_o(u)\, du + C \tag{55b}$$

where $C = 0$ since the integral for $z = 0$ must vanish according to the residue theorem. Hence (54) becomes (45), as had to be demonstrated.

(b) General Case of the Downwash Distribution on the Harmonically Vibrating Wing

In this section we assume an arbitrary downwash distribution of the form

$$w(x,t) = \frac{\partial y_o}{\partial t} + U \frac{\partial y_o}{\partial x} = v_o(x)\, e^{-i\omega t} \tag{56}$$

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[14] Courant and Hilbert, Methoden der mathematischen Physik I, (verlag J. Springer), p. 390.

with, however, the distinct requirement that the (infinitesimal) displacement y_o of the mean camber line from its zero position (x-axis) must everywhere be continuous and especially for $x = 0$ must change continuously into $y_o = 0$. For $x < 0$, we assume $y_o = 0$, and therefore $v_o(x) = 0$. It is easy to allow for discontinuities of $\frac{\partial y_o}{\partial x}$ by passing to the limit of a sequence of continuous functions y_o (compare case c).

For the general case of an arbitrary downwash distribution the right side of the integral equation need not be further specified; also, in the equations corresponding to (37) and (39) the same general expression appears on the right side as in (12). If we again introduce the dimensionless variable $z = \frac{kx}{\epsilon}$ instead of x and denote $v_o(x)$ as a function of z by $\gamma(z)$, we obtain

$$\frac{dv_o(x)}{dx} - i\frac{k}{M}v_o = \frac{k}{\epsilon}\frac{d\gamma(z)}{dz} - i\frac{k}{M}\gamma(z)$$

The form of the integro-differential equation for $F(z) \equiv f(x)$ which corresponds to (44) becomes, therefore, for positive values of z (after again dividing by a factor k)

$$-iMF(z) + F'(z) + \int_0^z G(z - \zeta)\, F(\zeta)\, d\zeta = \frac{1}{\epsilon} r(z) \tag{57}$$

with the abbreviation

$$r(z) = \frac{d\gamma(z)}{dz} - i\frac{\epsilon}{M}\gamma(z) \tag{57a}$$

Equation (57) has to be solved subject to the boundary condition $F(0) = 0$, as can be inferred from the assumed continuity of y_o at $x = 0$; for $z < 0$, $F(z) = 0$.

If we apply the Laplace transformation to (57) and denote the Laplace transform of $F(z)$ by $\psi(s)$, we obtain

$$\left[s - iM + g(s)\right]\ \psi(s) = \frac{1}{\epsilon} L_s \left[r(z)\right]$$

If we carry out the inverse transformation $\psi(s) \longrightarrow F(z)$, we obtain, taking (51) into consideration, according to definition

$$F(z) = \frac{1}{\epsilon} L_z^{-1} \left\{ \frac{1}{\sqrt{(s - iM)^2 + 1}} \; L_s \left[r(z) \right] \right\} \tag{58}$$

If we now consider that, according to (53) and (55)

$$L_z^{-1} \left[\frac{1}{\sqrt{(s - iM)^2 + 1}} \right] = e^{iMz} J_o(z) \equiv K(z) \tag{59}$$

and furthermore, that the result of the inverse transformation of the product of two functions (in s) is the convolution of the inversely transformed factors (in z), there immediately follows from (58) and (59)

$$F(z) = \frac{1}{\epsilon} \int_0^z e^{iM(z-\zeta)} J_o(z - \zeta) \, r(\zeta) \, d\zeta \tag{60}$$

which represents the solution of equation (57). By inserting (60) into (48) one then obtains for the pressure jump on the wing (for $z \geqq 0$)

$$\pi = -2\rho_o U \frac{1}{\sqrt{\epsilon}} \int_0^z K(z - \zeta) \, r(\zeta) \, d\zeta \tag{61}$$

with the meaning of $K(z)$ explained by (59)

(c) Transformation of the Results (60) and (61)

We at first confirm that the special formulas (46) and (49) for bending oscillations as a limiting case are contained in our general results (60) and (61). We have to make allowance for $\gamma(z)$ jumping near $z = 0$ practically discontinuously from 0 to the constant final value $a\epsilon = -i\omega A$ so that in (60) the term $\frac{d\gamma(\zeta)}{d\zeta}$ in $r(\zeta)$, equation (57a), makes a contribution only at the point $\zeta = 0$ and there

can be represented by

$$\frac{d\gamma(\xi)}{d\xi} = a\epsilon\,\delta(\xi - 0)$$

Thus we obtain according to (15)

$$\int_0^z K(z - \xi)\,\frac{d\gamma(\xi)}{d\xi}\,d\xi = a\epsilon\int_0^z K(z - \xi)\,\delta(\xi - 0)\,d\xi = a\epsilon\,K(z)$$

F(z) therefore becomes, according to (57a) and (60), in agreement with (46)

$$F(z) = aK(z) - ai\frac{\epsilon}{M}\int_0^z K(z - \xi)\,d\xi = a\left[K(z) - i\frac{\epsilon}{M}\int_0^z K(u)\,d\right]$$

In order to recognize that the general case also can be traced back to the type of bending oscillations we perform an integration by parts on (60), introducing temporarily the function

$$\overline{K}(z) = \int_0^z e^{iMu}J_0(u)\,du \quad (\text{for } z \gtrless 0) \tag{62}$$

Thus we find

$$\int_0^z K(z - \xi)\,\gamma(\xi)\,d\xi = \overline{K}(z - \xi)\,\gamma(\xi)\Big|_0^z + \int_0^z \overline{K}(z - \xi)\,\frac{d\gamma(\xi)}{d\xi}\,d\xi$$

The part to be evaluated between the limits vanished, since $\gamma(0) = 0$ and $\overline{K}(0) = 0$. Therefore, (60) may be transformed according to (57a):

$$F(z) = \frac{1}{\epsilon}\int_0^z\left[K(z - \xi) - i\frac{\epsilon}{M}\,\overline{K}(z - \xi)\right]\frac{d\gamma(\xi)}{d\xi}\,d\xi$$

and from (46) and (62):

$$F(z) = \frac{1}{\epsilon} \int_0^z H(z - \zeta) \frac{d\gamma(\zeta)}{d\zeta} d\zeta \tag{63}$$

Only the influence function $H(z)$, which is characteristic for the bending oscillations, appears in (63) besides $d\gamma/d\zeta$. Thus the pressure distribution appears composed of the integrated effect of infinitesimal steps, the magnitude of which is proportional to the derivative $d\gamma/dz$.

For practical needs it will, in general, be more useful to eliminate the derivative $d\gamma/d\zeta$ in (60) by integration by parts and to trace the pressure distribution back to an integral with respect to $\gamma(\zeta)$. We find

$$\int_0^z K(z - \zeta) \frac{d\gamma}{d\zeta} d\zeta = \gamma(z) + \int_0^z K'(-\zeta)\ \gamma(\zeta)\ d\zeta$$

since $\gamma(0) = 0$ and $K(0) = 1$. If we take the derivative of K we obtain from (59) and (60) after an easy transformation for $z \geqq 0$

$$F(z) = \frac{1}{\epsilon} \gamma(z) + \frac{1}{\epsilon} \int_0^z e^{iM(z-\zeta)} \left[\frac{i}{M} J_0(z - \zeta) - J_1(z - \zeta)\right] \gamma(\zeta)\ d\zeta \tag{64}$$

In this formula there appears nothing but the downwash distribution $\gamma(z)$ and a characteristic influence function.

The final formulas (60) to (64) are in complete agreement with the results of Mr. L. Schwarz[15], as one can easily confirm after an adequate change of the symbols. Therefore, further detailed statements with respect to the application of our formulas to the calculation of the air forces for vibrating lifting surfaces are ommitted since these calculations would be essentially identical with those by Schwarz.

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[15] L. Schwarz, elsewhere.

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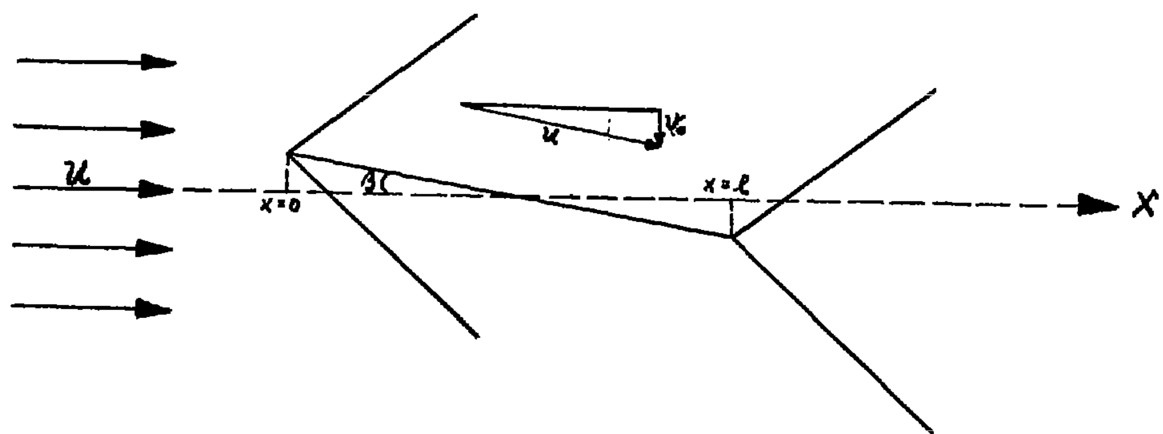

Figure 1.- Plate in supersonic flow.

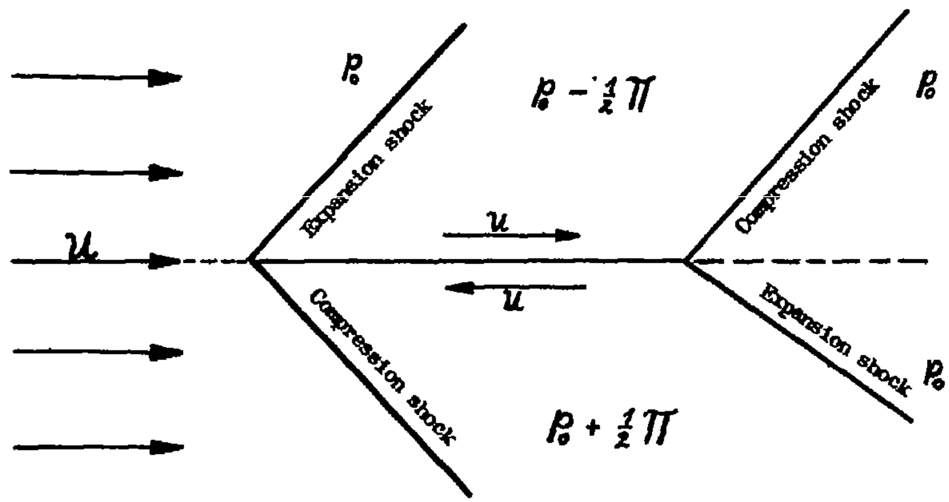

Figure 2.- Pressure distribution on a plate in supersonic flow. (p_0 normal pressure, π pressure drop.)